Compact-Reconstruction Weighted Essentially Non-Oscillatory Schemes for Hyperbolic Conservation Laws

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Doctoral Dissertation Defense
Applied Mathematics & Statistics, and Scientific Computation

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Motivation

Numerical Solution of Compressible Turbulent Flows
- Aircraft and Rotorcraft wake flows
- Characterized by large range of length scales
- Convection and interaction of eddies
- Compressibility → Shock waves & Shocklets
- Thin shear layers → High gradients in flow

High order accurate Navier-Stokes solver
- High spectral resolution for accurate capturing of smaller length scales
- Non-oscillatory solution across shock waves and shear layers
- Low dissipation errors for preservation of flow structures over large distances
Outline

• Hyperbolic Conservation Laws (Introduction / Background)
  – Numerical solution (Reconstruction + Time – marching)
  – High-resolution schemes in literature (Previous work)
  – Objectives of this thesis

• Weighted Non – Linear Compact Schemes
  – Derivation & analysis of 5th order Compact-Reconstruction WENO (CRWENO) schemes
  – Application to scalar conservation laws (accuracy, order of convergence, resolution)
  – Numerical cost + computational efficiency (compared to non-compact WENO schemes)
  – Alternative formulations for the non-linear weights

• Application to the Inviscid Euler Equations
  – Reconstruction of conserved, primitive and characteristic variables
  – Benchmark 1D and 2D inviscid flow problems (accuracy, convergence, resolution)
  – Numerical cost (conserved/primitive vs. characteristic reconstruction)

• Integration with a finite volume Navier-Stokes solver
  – Validation/verification for curvilinear, overset grids with relative motion → Application to flows around airfoils, wings, helicopter rotor blades
  – Direct numerical simulation of compressible, turbulent flows

• Conclusions and Future Work
Introduction / Background
Hyperbolic Conservation Laws

Scalar hyperbolic partial differential equation

\[ u_t + f(u)_x = 0; \quad f'(u) \in \mathcal{R} \]

Conservative discretization in space leads to an ordinary differential equation in time (solved by explicit / implicit ODE solvers)

\[
\frac{du_j}{dt} + \frac{1}{\Delta x} \left[ f(x_{j+1/2},t) - f(x_{j-1/2},t) \right] = 0
\]

Euler Explicit (1st order)
TVD-RK3 (3rd order)
BDF2 (Implicit 2nd order)
Reconstruction of Interface Flux

**Reconstruction** – interpolation of $f$ at the interfaces from the cell centered / averaged values (Focus of this thesis)

$$f_{j+1/2} = f\left(f_i : i = j - m, \ldots, j + n\right)$$

**Interpolation Stencil**

**Upwinding** – biased interpolation stencil to model wave nature of the solution

$$f_{j+1/2} = \begin{cases} f^L_{j+1/2} & \text{if } f'(u)\Big|_{x=x_{j+1/2}} > 0; \quad (m \geq n) \\ f^R_{j+1/2} & \text{if } f'(u)\Big|_{x=x_{j+1/2}} < 0; \quad (m < n) \end{cases}$$

- **Left – biased**
- **Right – biased**
The Weighted Essentially Non-Oscillatory (WENO) schemes

- Liu, Osher & Chan (JCP, 1994) and Jiang & Shu (JCP, 1996)
- At each interface, \( r \) possible \( r \)-th order interpolation schemes
- Final interface flux = convex combination of the \( r \)-th order interpolations
- Optimal weights in smooth regions \( \rightarrow (2r-1) \)-th order interpolation
- Smoothness – dependent weights for discontinuous solutions
  \( \rightarrow \) Non-oscillatory interpolation

\[
f_{j+1/2} = \sum_{k=1}^{r} \omega_k f_{j+1/2}^k
\]

\[
\alpha_k = \frac{c_k}{(\beta_k + \varepsilon)^p}; \quad \omega_k = \frac{\alpha_k}{\sum_k \alpha_k}
\]

Optimal Weights

WENO Weights

Smoothness Indicators
5th Order WENO scheme

\[ f_{j+1/2} = \frac{1}{3} f_{j-2} - \frac{7}{6} f_{j-1} + \frac{11}{6} f_j \]

\[ f_{j+1/2} = -\frac{1}{6} f_{j-1} + \frac{5}{6} f_j + \frac{1}{3} f_{j+1} \]

\[ f_{j+3/2} = \frac{1}{3} f_j + \frac{5}{6} f_{j+1} - \frac{1}{6} f_{j+2} \]

\[ f_{j+1/2} = \frac{\omega_1}{3} f_{j-2} + \frac{11}{306} f_{j-1} + \frac{13}{60} f_j + \frac{47}{60} f_{j+1} + \frac{27}{60} f_{j+2} + \frac{1}{20} \left( 2 \omega_2 + 5 \omega_3 \right) f_{j+1} - \frac{\omega_3}{6} f_{j+2} \]
Non-Linear Weights

Weights are calculated based on smoothness indicators

\[ \beta_1 = IS(f_{j-2}, f_{j-1}, f_j) \]
\[ \beta_2 = IS(f_{j-1}, f_j, f_{j+1}) \]
\[ \beta_3 = IS(f_j, f_{j+1}, f_{j+2}) \]

\[ \alpha_k = \begin{cases} 
\frac{c_k}{(\beta_k + \varepsilon)^p} \\
1 + \left( \frac{\tau}{\varepsilon + \beta_k} \right)^p 
\end{cases} \]

Implementations of WENO weights:
- Jiang & Shu (1996) \(\rightarrow\) WENO5-JS
- Henrick, Aslam & Powers (2005) \(\rightarrow\) WENO5-M
- Borges, et. al. (2008) \(\rightarrow\) WENO5-Z
- Yamaleev & Carpenter (2009) \(\rightarrow\) WENO5-YC
Compact Difference Schemes

Introduced by Lele (JCP, 1992) in non-conservative form

\[ u_t + f(u)_x = 0 \Rightarrow \frac{du_j}{dt} + \hat{f}_{x,j} = 0 \]  
(Non – conservative discretization)

\[ \beta \hat{f}_{x,j-2} + \alpha \hat{f}_{x,j-1} + \hat{f}_{x,j} + \alpha \hat{f}_{x,j+1} + \beta \hat{f}_{x,j+2} = a \frac{f_{j+1} - f_{j-1}}{2\Delta x} + b \frac{f_{j+2} - f_{j-2}}{4\Delta x} + c \frac{f_{j+3} - f_{j-3}}{3\Delta x} \]

Central differencing schemes
\[ \alpha = \beta = 0 \quad : \text{Non-compact} \]
\[ \beta = 0 \quad : \text{Tridiagonal compact} \]
\[ \alpha \neq 0, \beta \neq 0 \quad : \text{Penta-diagonal compact} \]

- Coupling between neighboring flux derivative approximations
- Requires the solution of a tridiagonal / penta-diagonal system of equations (sparse LU)
- Constant coefficients \( \Rightarrow \) Pre-computed LU decomposition
Compact schemes have higher spectral resolution for same order of convergence ➔ Application to problems with large range of length scales
Previous Work

Compressible Turbulent Flows

Non-Oscillatory (WENO Schemes) + High spectral Resolution (Compact schemes)

Hybrid Compact-WENO Schemes

- Smoothness indicator marks out regions of near discontinuities
- Compact scheme for smooth regions
- WENO scheme near discontinuities

Disadvantages:

- Loss of spectral resolution near discontinuities
- Requirement of an arbitrary parameter for switching between schemes
- Presence of shocklets throughout the domain → Use WENO everywhere?

Isotropic Turbulence Decay
Previous Work

Compressible Turbulent Flows

Non-Oscillatory (WENO Schemes) + High spectral Resolution (Compact schemes)

Weighted Compact Non-Linear Schemes (WCNS):
• Non-conservative finite difference discretization for a staggered mesh arrangement
• Step 1: $f_{j+1/2}$ computed from $f_j$ using the WENO scheme (Non-oscillatory)
• Step 2: $f_x$ computed from $f_{j+1/2}$ using high order central compact difference scheme

Disadvantage: Marginal increase in spectral resolution (due to reconstruction of interface flux with WENO scheme)

Objectives

• Derivation and implementation of CRWENO schemes
  – Formulation: Identification of lower-order compact interpolation schemes & application of WENO weights
  – Numerical analysis of underlying linear schemes
  – Formulation and analysis of boundary closure
  – Application to scalar conservation laws: Study accuracy, convergence, non-oscillatory behavior, and computational efficiency

• Extension to system of equation: inviscid Euler equations
  – Application to vector quantities: conserved, primitive & characteristic variables
  – Numerical properties: assessed on benchmark inviscid flow problems

• Integration with a finite-volume Navier-Stokes solver
  – Application to steady and unsteady flows around airfoils, wings and rotors: Comparison of the resolution for near-blade and wake flow features
  – Direct numerical simulation of canonical compressible, turbulent flows
Weighted Non-Linear Compact Schemes
CRWENO Schemes

- General form of a conservative compact scheme:
  \[ A(\hat{f}_{j+1/2-m}, \ldots, \hat{f}_{j+1/2}, \ldots, \hat{f}_{j+1/2+m}) = B(f_{j-n}, \ldots, f_j, \ldots, f_{j+n}) \quad \Rightarrow \quad [A]\hat{f} = [B]f \]

- At each interface, \( r \) possible \((r)\)-th order compact interpolations, combined using optimal weights \( c_k \) to yield \((2r-1)\)-th order compact interpolation scheme:
  \[
  \sum_{k=1}^{r} c_k A_k^r(\hat{f}_{j+1/2-m}, \ldots, \hat{f}_{j+1/2+m}) = \sum_{k=1}^{r} c_k B_k^r(f_{j-n}, \ldots, f_{j+n}) \Rightarrow A^{2r-1}(\hat{f}_{j+1/2-m}, \ldots, \hat{f}_{j+1/2+m}) = B^{2r-1}(f_{j-n}, \ldots, f_{j+n})
  \]

- Apply WENO algorithm on the optimal weights \( c_k \) – scale them according to local smoothness
  \[
  \sum_{k=1}^{r} \omega_k A_k^r(\hat{f}_{j+1/2-m}, \ldots, \hat{f}_{j+1/2+m}) = \sum_{k=1}^{r} \omega_k B_k^r(f_{j-n}, \ldots, f_{j+n}) \quad \Rightarrow \quad \alpha_k = \frac{c_k}{(\beta_k + \varepsilon)^p}; \quad \omega_k = \alpha_k / \sum_k \alpha_k
  \]
The 5th Order CRWENO scheme (CRWENO5) is shown in the figure. The scheme is defined by the following equations:

\[
\frac{2}{3} f_{j-1/2} + \frac{1}{3} f_{j+1/2} = \frac{1}{6} f_{j-1} + \frac{5}{6} f_j
\]

\[
\frac{1}{3} f_{j-1/2} + \frac{2}{3} f_{j+1/2} = \frac{5}{6} f_j + \frac{1}{6} f_{j+1}
\]

\[
\frac{2}{3} f_{j+1/2} + \frac{1}{3} f_{j+3/2} = \frac{1}{6} f_j + \frac{5}{6} f_{j+1}
\]

The weights \(\omega_1\), \(\omega_2\), and \(\omega_3\) are given by:

\[
c_1 = \frac{2}{10}
\]

\[
c_2 = \frac{5}{10}
\]

\[
c_3 = \frac{3}{10}
\]
Low Dissipation 5th order CRWENO scheme (CRWENO5-LD)

\[
\begin{align*}
\frac{2}{3} f_{j-1/2} + \frac{1}{3} f_{j+1/2} &= \frac{1}{6} f_{j-1} + \frac{5}{6} f_j \\
\frac{1}{3} f_{j-1/2} + \frac{2}{3} f_{j+1/2} &= \frac{5}{6} f_j + \frac{1}{6} f_{j+1} \\
\frac{2}{3} f_{j+1/2} + \frac{1}{3} f_{j+3/2} &= \frac{1}{6} f_{j} + \frac{5}{6} f_{j+1} \\
\frac{1}{3} f_{j+1/2} + \frac{2}{3} f_{j+3/2} &= \frac{5}{6} f_{j+1} + \frac{1}{6} f_{j+2}
\end{align*}
\]

\[
\left( \frac{2}{3} \omega_1 \frac{5}{20} f_j \right) f_{j-1/2} + \left( \frac{1}{3} \omega_2 \frac{2}{20} f_j + \frac{3}{20} \omega_3 f_{j+1/2} \right) f_j + \left( \frac{2}{3} \omega_4 \frac{1}{20} f_{j+1/2} \right) f_{j+3/2} = \frac{3}{120} \left( \frac{1}{3} \omega_3 \frac{4}{3} f_{j-1/2} \right) f_{j-1} + \frac{49}{6120} f_{j-1/2} + \frac{1}{6} f_{j+1/2} + \frac{\omega_3}{6} f_{j+3/2} + 5(\omega_3 + \omega_4) f_{j+3/2} \right) + \frac{\omega_1}{6} f_{j+1/2} + \frac{\omega_2}{6} f_{j+1/2} + \frac{\omega_4}{6} f_{j+1/2}
\]
Smoothness Indicators

Weights are calculated based on smoothness indicators of corresponding explicit stencils (same as WENO5 scheme)

\[
\begin{align*}
\beta_1 &= IS(f_{j-2}, f_{j-1}, f_j) \\
\beta_2 &= IS(f_{j-1}, f_j, f_{j+1}) \\
\beta_3 &= IS(f_j, f_{j+1}, f_{j+2}) \\
\beta_4 &= IS(f_{j+1}, f_{j+2}, f_{j+3})
\end{align*}
\]

\[
\alpha_k = \frac{c_k}{(\beta_k + \varepsilon)^p}; \quad \omega_k = \frac{\alpha_k}{\sum_k \alpha_k}; \quad k = 1, 2, 3
\]

\[
\beta_4 = \max(\beta_3, \beta_4) \quad \text{(Avoid downwind interpolation)}
\]
Behavior across Discontinuities

Decoupling of domain across a discontinuity

\[
\begin{align*}
\frac{2}{3} f_{j-1/2} + \frac{1}{3} f_{j+1/2} &= \frac{1}{6} f_{j-1} + \frac{5}{6} f_j \quad \omega_1 \quad \text{✓} \quad \text{✓} \quad 0 \quad 0 \\
\frac{1}{3} f_{j-1/2} + \frac{2}{3} f_{j+1/2} &= \frac{5}{6} f_j + \frac{1}{6} f_{j+1} \quad \omega_2 \quad \text{✓} \quad 0 \quad 0 \\
\frac{2}{3} f_{j+1/2} + \frac{1}{3} f_{j+3/2} &= \frac{1}{6} f_j + \frac{5}{6} f_{j+1} \quad \omega_3 \quad 0 \quad 0 \quad \text{✓} \\
\end{align*}
\]
Numerical Analysis

Underlying optimal (linear) schemes:

\[
f_{j+1/2} = \frac{1}{30} f_{j-2} - \frac{13}{60} f_{j-1} + \frac{47}{60} f_j + \frac{27}{60} f_{j+1} - \frac{1}{20} f_{j+2}
\]

\[
\frac{3}{10} f_{j-1/2} + \frac{6}{10} f_{j+1/2} + \frac{1}{10} f_{j+3/2} = \frac{1}{30} f_{j-1} + \frac{19}{30} f_j + \frac{10}{30} f_{j+1}
\]

\[
\frac{5}{20} f_{j-1/2} + \frac{12}{20} f_{j+1/2} + \frac{3}{20} f_{j+3/2} = \frac{3}{120} f_{j-1} + \frac{67}{120} f_j + \frac{49}{120} f_{j+1} + \frac{1}{120} f_{j+2}
\]

Taylor Series Analysis: Dissipation and dispersion errors

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Dissipation</th>
<th>Dispersion</th>
</tr>
</thead>
<tbody>
<tr>
<td>WENO5</td>
<td>(\frac{1}{60} \frac{\partial^6 f}{\partial x^6} \bigg</td>
<td>\Delta x^5)</td>
</tr>
<tr>
<td>CRWENO5</td>
<td>(\frac{1}{600} \frac{\partial^6 f}{\partial x^6} \bigg</td>
<td>\Delta x^5)</td>
</tr>
<tr>
<td>CRWENO5-LD</td>
<td>(\frac{1}{1200} \frac{\partial^6 f}{\partial x^6} \bigg</td>
<td>\Delta x^5)</td>
</tr>
</tbody>
</table>
Numerical Analysis

Fourier Analysis: spectral properties

- Exact: (-)
- First Order: (…)
- WENO5
- WENO9
- CRWENO5
- CRWENO5-LD

Modified Phase $k\Delta x$

Phase $k\Delta x$

Normalized Phase Error $\epsilon_k$

Dissipation $\sigma$

Normalized Phase $k\Delta x/\pi$
Comparison of Spectral Resolutions

Comparison of spectral resolution and bandwidth resolving efficiency — CRWENO5 scheme with high-resolution schemes in literature

Bandwidth Resolving Efficiency

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Efficiency</th>
</tr>
</thead>
<tbody>
<tr>
<td>WENO5 (Jiang &amp; Shu, 1996)</td>
<td>0.35</td>
</tr>
<tr>
<td>WENO7 (Balsara &amp; Shu, 2000)</td>
<td>0.42</td>
</tr>
<tr>
<td>WENO9 (Balsara &amp; Shu, 2000)</td>
<td>0.48</td>
</tr>
<tr>
<td>CRWENO5</td>
<td>0.61</td>
</tr>
<tr>
<td>CRWENO5-LD</td>
<td>0.52</td>
</tr>
<tr>
<td>6th-order central compact (Lele, 1992)</td>
<td>0.50</td>
</tr>
<tr>
<td>8th-order central compact (Lele, 1992)</td>
<td>0.58</td>
</tr>
<tr>
<td>WENO-SYMBO (r = 3) (Martin, et. al., 2006)</td>
<td>0.49</td>
</tr>
<tr>
<td>WENO-SYMBO (r = 4) (Martin, et. al., 2006)</td>
<td>0.56</td>
</tr>
</tbody>
</table>

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i. CRWENO5
ii. CRWENO5-LD
iii. 6th order central compact (Lele, 1992)
iv. 8th order central compact (Lele, 1992)
v. WENO-SYMBO (r=3) (Martin, et. al., 2006)
vi. WENO-SYMBO (r=4) (Martin, et. al., 2006)
vii. WCNS5 (Deng & Zhang, 2000)
Boundary Closure

Cell interface aligned with physical boundary – Boundary conditions implemented on “ghost cells”

Cell center coincident on physical boundary – Reduced-order / interior-biased interpolation scheme at interfaces near the boundary
Scalar Conservation Laws

Schemes validated for the linear advection equation and the inviscid Burgers equation

Smooth problems

- 5th order convergence verified for the new schemes
- Errors for CRWENO5 order of magnitude lower than WENO5, errors for CRWENO5-LD half those of CRWENO5

Linear advection equation

\[ u_0(x) = \sin(2\pi x); \ 0 \leq x \leq 1 \]

(Periodic domain)

Solution obtained after 1 cycle with TVD-RK3 time-stepping
Scalar Conservation Laws

Schemes validated for the **linear advection equation** and the **inviscid Burgers equation**

Discontinuous problems
- Non-oscillatory behavior validated across discontinuities
- CRWENO schemes show better resolution of discontinuous data (lower smearing and clipping)
Computational Efficiency

- CRWENO schemes require solution to tridiagonal system at each iteration (solution-dependent weights) ⇒ Higher numerical cost for same number of points
- Higher accuracy (error is 1/10th that of WENO) ⇒ Comparable solution obtained on coarser mesh ⇒ Computationally more efficient

Error vs. runtime for smooth problem

Periodic advection of a triangular wave after 100 cycles (120 point mesh)
Implementation of Non-Linear Weights

• Definition of the WENO weights by Jiang & Shu (1996):

\[ \alpha_k = \frac{c_k}{(\beta_k + \varepsilon)^p} \quad p = 2 \quad \varepsilon = 10^{-6} \]

- \(\varepsilon\) – sensitivity: Lower values of \(\varepsilon\) (10^{-40}) results in sub-optimal convergence for smooth problems with critical points (vanishing derivatives)
- Excessive dissipation: Original weights defined by Jiang & Shu were found to be too dissipative across discontinuities

• Alternative formulations proposed in literature:
  - Mapped WENO (Henrick, Aslam & Powers, 2005): Defined a mapping function that improved convergence to WENO weights

\[ g_k(\omega) = \frac{\omega \left( c_k + c_k^2 - 3c_k \omega + \omega^2 \right)}{c_k^2 + \omega \left( 1 - 2c_k \right)} \]

- WENO-Z (Borges, et. al., 2008)
- ESWENO (Yamaleev & Carpenter, 2009)

\[ \alpha_k = c_k \left[ 1 + \left( \frac{\tau}{\varepsilon + \beta_k} \right)^p \right] \]

• These improved convergence and resolution of the WENO schemes
Implementation of Non-Linear Weights

• Improved accuracy and convergence for smooth problems
• Sharper resolution for extrema and discontinuities
• Yamaleev & Carpenter weights show non-oscillatory behavior for higher derivatives (for this case)

Solution after 1 cycle

\[ u_0(x) = \sin\left(\pi x - \frac{\sin(\pi x)}{\pi}\right) \]

\[-1 \leq x \leq 1 \text{ (Periodic)}\]

Solutions of exponential and square waves after 50 cycles

CRWENO5-JS
CRWENO5-M
CRWENO5-Z
CRWENO5-YC
Application to Inviscid Euler Equations
Euler Equations

- Compressible Euler equations in 1D given by

\[
\begin{bmatrix}
\frac{\partial \rho}{\partial t} \\
\frac{\partial \rho u}{\partial t} \\
\frac{\partial e}{\partial t}
\end{bmatrix} + \begin{bmatrix}
\frac{\partial \rho u}{\partial x} \\
\frac{\partial \rho u^2 + p}{\partial x} \\
\frac{(e + p)u}{\partial x}
\end{bmatrix} = 0
\]

\( \rho = \) Density
\( u = \) Velocity
\( p = \) Pressure
\( e = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2 \) (Internal energy)

- In the form of a general hyperbolic PDE

\[ u_t + f(u)_x = 0 \]

- Extension of scalar interpolation schemes to a system of equations
  - Component-wise reconstruction of conserved variables \((\rho, \rho u, e)\)
  - Reconstruction of primitive (flow) variables \((\rho, u, p)\)
  - Reconstruction of characteristic variables

- Extension to conserved/primitive variables trivial
  - Solution of \(D+2\) tridiagonal systems of equations (\(D\) is the number of dimensions)
CRWENO for Euler Equations

Characteristic based reconstruction respects the physics of the problem – 1D scalar wave propagation along each characteristic

\[
\begin{align*}
    a \alpha^k_{j-1/2} + b \alpha^k_{j+1/2} + c \alpha^k_{j+3/2} &= \tilde{a} \alpha^k_{j-1} + \tilde{b} \alpha^k_j + \tilde{c} \alpha^k_{j+1} \\
    (\alpha^k_i &= l^k_{j+1/2} \cdot f) \\
    \lambda^k_{j+1/2}, l^k_{j+1/2}, r^k_{j+1/2} & \text{Eigenvalues, left and right eigenvectors}
\end{align*}
\]

U_{avg} (Roe averaged)

\[
\begin{align*}
    a(l^{k1}_{j+1/2} f^{1}_{j-1/2} + l^{k2}_{j+1/2} f^{2}_{j-1/2} + l^{k3}_{j+1/2} f^{3}_{j-1/2}) \\
    + b(l^{k1}_{j+1/2} f^{1}_{j+1/2} + l^{k2}_{j+1/2} f^{2}_{j+1/2} + l^{k3}_{j+1/2} f^{3}_{j+1/2}) \\
    + c(l^{k1}_{j+1/2} f^{1}_{j+3/2} + l^{k2}_{j+1/2} f^{2}_{j+3/2} + l^{k3}_{j+1/2} f^{3}_{j+3/2}) &= \tilde{a} \alpha^k_{j-1} + \tilde{b} \alpha^k_j + \tilde{c} \alpha^k_{j+1}
\end{align*}
\]

• Results in a block tri-diagonal linear system along each dimension (as compared to tri-diagonal system for conserved/primitive reconstruction)
• For multi-dimensions, solution of linear system required along each grid line
• Upwinding –
  – Left and right biased fluxes computed
  – Roe-Fixed formulation used

\[
\alpha^k_{j+1/2}, \alpha^k_{j+1/2} \Rightarrow \alpha^k_{j+1/2}
\]
Entropy Wave Advection

- **Accuracy** – CRWENO5 errors 1/10th that of WENO5
- **Smooth problem** – Reconstruction of conserved, primitive and characteristic yield identical solutions
- **Computational expense** - CRWENO5 is more efficient for conserved/primitive reconstruction, but not for characteristic reconstruction
Shock – Entropy Interaction

- Interaction of a shock wave with a density wave resulting in high-frequency waves and discontinuities
- CRWENO scheme shows better resolution of high-resolution waves than WENO5
- Further improvement by using the alternative formulations for the WENO weights

\[ \varepsilon = 10^{-6}, p = 1 \]
Isentropic Vortex Convection

Solution after travelling 1000 core radii
Compact schemes show better shape and strength preservation for long term convection

Initial

WENO5

CRWENO5

CRWENO5-LD

NonCompact5 (16.5)
Compact5 (20.48)
Compact5-LD (20.56)
WENO5 (27.88)
CRWENO5 (33.38)
CRWENO5-LD (38.70)
WENO5 - 90x90 grid (92.88)
Isentropic Vortex Convection

Long term inviscid convection (1000 core radii) – Preservation of strength and shape

Density Contours

<table>
<thead>
<tr>
<th>Scheme</th>
<th>Numerical Shadowgraph (Laplacian of density)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No Limiting (5th order compact)</td>
<td>Smooth, non-oscillatory density field for all schemes (almost identical)</td>
</tr>
<tr>
<td>CRWENO5-JS</td>
<td>Oscillations in 2nd derivative for CRWENO5-JS &amp; CRWENO5-Mapped, while CRWENO5-YC and CRWENO5-Borges are smooth</td>
</tr>
</tbody>
</table>

\[ \varepsilon = 10^{-6}, \ p = 2 \]
Double Mach Reflection of a Mach 10 shock on a 480 X 120 grid

- CRWENO schemes validation for 2D problem with strong discontinuities
- Better capturing of the contact discontinuity roll-up

720 X 180 grid
Shock – Vorticity Interaction

\[ \theta = \pi/6 \]
(Angle of vorticity wave)

Interaction of a shock with a vorticity wave:
- Accurate capturing of acoustic, vorticity and entropy waves
- Solutions obtained on 96x64 and 192x128 grids
- CRWENO5 shows reduced clipping of the waves at both grid resolutions
Integration with a Finite-Volume Navier-Stokes Solver
Baseline Solver

Integration of the CRWENO scheme with a compressible Navier Stokes solver for overset structured meshes

- **Time Marching:** 2\textsuperscript{nd} order Backward Differencing (BDF2) and 3\textsuperscript{rd} order Total Variation Diminishing Runge Kutta (TVDRK3)
- **Dual time-stepping** for time-accurate computations
- **Implicit Inversion:** Diagonalized ADI and LU-SGS
- **Spatial reconstruction:**
  - 5\textsuperscript{th} order CRWENO scheme (compact)
  - 3\textsuperscript{rd} order MUSCL and 5\textsuperscript{th} order WENO schemes (non-compact)
- **Upwinding:** Roe’s flux differencing
- **Turbulence Modeling:** Spallart-Almaras one-equation model
- **Implicit hole-cutting** for overset meshes
- **Viscous Terms** discretized by 2\textsuperscript{nd} order central differences
Steady Flow around RAE2822 Airfoil

Turbulent, transonic flow around RAE2822

Flow Conditions:
Reynolds number 6.5 million, angle of attack 2.51°, freestream Mach number 0.731

Numerical Solution:
Grid: 521x171 C-type
Outer boundary: 50c
CRWENO5 in space
BDF2 in time

Validation w/ expt. results

Conserved variable reconstruction leads to poor convergence!
Pitching – Plunging NACA0005

Flow Conditions
Reynolds No. = 15000, Freestream Mach = 0.1
Pitch amplitude = 40°, Plunge amplitude = 1.0
Reduced frequency = 0.795

Grid dimensions: 391x161
Time stepping: BDF2 w/ 15 sub-iterations

Lift and drag over one cycle
Pitching – Plunging NACA0005

Numerical Shadowgraph at $t/T = 0.75$

Out-of-plane Vorticity at $t/T = 0.40$

CRWENO5 shows improvement in capturing acoustic waves and vortical structures
ONERA-M6 Wing

- Steady, transonic flow around the ONERA-M6 wing
- C-O mesh with 289x65x49 points
- Freestream Re = 21.67 million
- Freestream Mach number = 0.84
- Angle of attack = 3.06°
- Surface pressure coefficient validated with experimental data
ONERA-M6 Wing

CRWENO5 shows an improvement in the tip vortex resolution and preservation in the wake region
Overset Grids

Solution algorithm on overset meshes
- Identification of field, overlap and hole regions
- Field points → Governing equations are solved
- Overlap region → Solution exchanged with other meshes
- Hole region → Blanked out, contains non-physical values
- Implicit Hole-Cutting (Lee & Baeder, 2008)
- Tri-linear interpolation of solution between donor and receiver points

Application of compact schemes
- Coupled solution for the interface fluxes
- Solution in hole region coupled with solution at field points
- System of equations contain non-physical values from the hole region
CRWENO on Overset Grids

Behavior across discontinuity ↔ Behavior across hole cut
(Non-physical values appear as a discontinuity)

Adaptive stenciling of the CRWENO scheme

Decoupling of solution between field and hole points
Flow Conditions:
Reynolds number = 4.15e6
Freestream Mach = 0.283
Angle of Attack = 0°, 10°

Verification of CRWENO5 scheme with non-compact MUSCL3 and WENO5 schemes

Wind Tunnel Mesh – Clustered Cartesian, 151x101 points

Airfoil Mesh – C-type, 365x138 points

Slat Mesh – C-type, 317x97 points

Pressure contours (CRWENO5 w/ BDF2)

α = 0°

α = 10°
SC2110 Airfoil w/ Slat in Wind Tunnel

- High gradients in flow between slat and airfoil
- Overlap and solution transfer between slat and airfoil meshes
- Pressure contours from CRWENO5 compared with those from non-compact schemes
Flow Conditions:
Reynolds number: 3.92 million, Freestream Mach number 0.302
Mean angle of attack: 9.78°, Pitch Amplitude: 9.9°, Reduced Frequency: 0.099, Tunnel height: 5c

Numerical Solution: Time stepping: BDF2 w/ 15 sub-iterations
Validation for Overset Meshes w/ Grid Motion

- Pressure
- Vorticity Magnitude

MUSCL3
WENO5
CRWENO5

- Shed vortices from upper surface
  - Contour lines are continuous between airfoil and wind tunnel meshes
  - Shed vortices pass smoothly between the two domains
Harrington 2-Bladed Rotor

Blade Mesh – C-O type, 267x78x56 points

Flow Conditions:
- $M_{\text{tip}}$: 0.352
- $Re_{\text{tip}}$: 3.5 million

Cylindrical Back-ground Mesh
127 x 116 x 118 points

Validation of thrust & power coefficients and figure of merit

Rotor Geometry:
- Aspect Ratio – 8.33
- Airfoil section – NACA (t/c: 27.5% @ 0.2R, 15% @ R)
Harrington 2-Bladed Rotor
(Near-Blade and Wake Flowfield)
Flow involves energy transfer to smaller length scales

Grid-converged solutions obtained on $128^3$ grid (WENO5 & CRWENO5 agree)

CRWENO5 shows better resolution of intermediate and higher wavenumbers

Iso-surfaces of vorticity magnitude, colored by pressure

$M_t = 0.3$
$Re_\lambda = 50$
Shock – Turbulence Interaction

\( M_i = 0.3 \)  
\( Re_\lambda = 50 \)

- Inflow: Fluctuations from isotropic turbulence decay added to mean flow at Mach 2
- Interaction with a shock wave magnifies the turbulent fluctuations
- Problem solved on two grids: 64x32x32 and 128x64x64 points (uniform)
- CRWENO5 → Lower dissipation → Predicts higher levels of fluctuations on both grids
Pre- and post-shock energy spectra (CRWENO5)

- Interaction with a shock wave amplifies intermediate and higher wavenumbers.
- **CRWENO5 shows improved resolution of the smaller length scales** (for both grids).
Conclusions and Future Work
Conclusions

• Derivation and implementation of 5\textsuperscript{th} order CRWENO schemes
  – Higher accuracy: Lower absolute errors than 5\textsuperscript{th} order WENO scheme
  – Reduced clipping and smearing for discontinuities and extrema
  – Higher computational efficiency: Lower runtime than the WENO scheme for solutions with same accuracy and resolution

• Extension to system of equations: inviscid Euler equations
  – Implemented for reconstruction of conserved, primitive & characteristic variables
  – Higher accuracy and reduced smearing & clipping of discontinuities
  – Improved resolution of small-length scale waves (Shock-entropy wave interaction)
  – Improved preservation of vortex strength and shape (Isentropic vortex convection)
  – Higher computational efficiency for conserved/primitive reconstruction, \textbf{NOT} for characteristic based reconstruction

• Integration with a finite-volume Navier-Stokes solver
  – Applied to steady/unsteady flow past airfoils / wing / rotor
  – Validated for curvilinear grids + overset meshes with relative motion
  – Improved resolution of near-blade and wake flow features (vortical structures)
  – DNS of compressible turbulent flows: Lower dissipation of smaller length scales
Key Contributions

Introduction of the Compact-Reconstruction WENO scheme
- Applies the WENO algorithm (solution-dependent interpolation) to compact schemes
- High spectral resolution \(\Rightarrow\) Improved resolution of small length scales
- Non-oscillatory interpolation across discontinuities and steep gradients
- Higher accuracy \(\Rightarrow\) Preservation of flow features for long-term convection
- Applicable to overset meshes with no special treatment

Comparison with compact-WENO hybrid schemes
- No need for an additional switching parameter
- Does not revert to non-compact scheme (poor spectral resolution) near discontinuities
Future Work

• Improvements / extensions of the numerical scheme
  – Implementation of non-linear weights (convergence for stationary shock and steady flow around airfoils)
  – Derivation for non-uniform grids
  – Implementation of a 9\textsuperscript{th} order CRWENO scheme
  – Fine & medium grain parallelization issues
  – Use of monotonicity-preserving (MP) limits (Suresh & Huynh, 1997)

• Applications
  – High – resolution solutions to aircraft / rotorcraft wake flow and interaction with ground plane for rotorcraft operating IGE
  – Implementation for domains with immersed boundaries
  – DNS of shock – boundary layer interactions
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