Linearized Coulomb Collision Operator for Simulation of Interpenetrating Plasma Streams

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***Abstract*— We present an extension of a linearized Coulomb collision operator, previously used in several Eulerian kinetic codes for like-species collisions and unlike-species collisions in the case where the backgrounds about which the linearization is made all are in collisional equilibrium, to the situation of interpenetrating plasma streams. In the latter case, the backgrounds cannot be taken to be in equilibrium and a significant generalization is required. Our development is targeted toward the Eulerian kinetic plasma code LOKI, which evolves the Vlasov–Poisson or Vlasov–Maxwell system in a Cartesian "2 + 2-dimensional" phase space. The extended operator has been implemented in a test code, and results of both quantitative verification and qualitative "realizability" tests are presented.**

***Index Terms*— Plasma simulation, plasma transport processes.**

I. Introduction

INTERPENETRATING plasma streams are of interest in astrophysics and laboratory high-energy density (including inertial confinement) plasma experiments. Our goal is to develop a kinetic simulation capability that can model interactions, including turbulent interactions, between such plasma streams. The use of Eulerian codes for kinetic plasma problems, both laminar [1]–[4] and turbulent [5]–[7], [9]–[12], has grown in recent years. We have based our development on LOKI [10] which is an Eulerian kinetic plasma code that evolves the phase-space density ("distribution function") in a 4-D (two space + two velocity) phase space using conservative high-order finite-volume and finite-difference methods and explicit Runge–Kutta (RK) time stepping. LOKI has been used to investigate a variety of nonlinear kinetic plasma effects [13]–[16].

Coulomb collisions can act in several important ways in interpenetrating plasma streams. Direct effects include drag, heating (transfer of streaming energy into thermal energy), isotropization and thermalization of individual streams, and equilibration of the whole system. Coulomb collisions can also have indirect effects through modification of growth rates (including complete stabilization), frequencies, and mode structure of plasma instabilities, as well as smoothing and thermalization of fine-scale features in the distribution functions resulting from nonlinear evolution driven by instabilities.

Prior to this paper, LOKI had a like-species (intrastream) collision operator implementation based on linearization about Maxwellian backgrounds [17], [18], with a reduced (rank) "backreaction" operator that enforces energy and momentum conservation, maintains a version of the Boltzmann H-theorem and also Galilean invariance. A high-order conservative finite-difference discretization was used for the "forward" drag and diffusion terms [15]. An extension of this operator to interspecies collisions, in the situation where the background Maxwellians are in equilibrium with each other (i.e., have the same temperatures and no relative flow) was given by Kolesnikov *et al.* [19].

A key need for our development was a Coulomb collision operator that can represent both intrastream and interstream collisions. For collisions between multiple interpenetrating plasma streams, the background Maxwellians can no longer be taken to be in equilibrium, and a significant extension is needed. In this paper, we describe such an extension and its development.

While we could have pursued a direct discretization of the bilinear operator using solutions of Rosenbluth's potential equations [20] with the actual distribution functions as sources to obtain the drag and diffusion coefficients, we have chosen to extend the existing linearized approach. This extension can make use of verified collision operator coding already present in LOKI and in a previously written MATLAB-based testbed code. We also believe that it provides a useful alternative method that has some desirable properties. The Rosenbluth potentials are functions of only the magnitude of a relative velocity and can, therefore, be computed using a 1-D lookup table. This may offer some computational speedup relative to the direct Rosenbluth potential computation approach, although any such speedups remain to be quantified. It also addresses a significant issue for LOKI implementation concerning the mixing of 2-D and 3-D velocity spaces that would also be present in the direct Rosenbluth potential calculation approach.

The outline of the remainder of this paper is as follows. In Section II, we review the Landau–Fokker–Planck collision

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operator, establish the key notation, and give the very useful Rosenbluth–Macdonald–Judd form. Section III discusses the linearization of the operator and the particular case of linearization about a Maxwellian background. The extension of the operator to the case of multiple plasma streams, which are inherently not in collisional equilibrium, is discussed in Section IV. Section V discusses specific issues related to the LOKI 2-D Cartesian velocity space, the generalization of the reduced backreaction operator to the nonequilibrium case of multiple plasma streams and the choices of the moments that define the background Maxwellians. Verification of the operator, including results for a "multiproperty" verification case that has been developed from many simpler cases that each check an individual property of the operator, is discussed in Section VI. Finally, Section VII gives a summary, conclusions, and a discussion of future near-term work.

II. Landau–Fokker–Planck Collision Operator

The Vlasov–Fokker–Planck equation, which describes a weakly coupled plasma charged species a, is [21]

$$\frac{\partial f_a}{\partial t}+\boldsymbol{v}\cdot\frac{\partial f_a}{\partial \boldsymbol{x}}+\boldsymbol{A}(\boldsymbol{x},\boldsymbol{v},t)\cdot\frac{\partial f_a}{\partial \boldsymbol{v}}=\left.\frac{\partial f_a}{\partial t}\right|_c=-\sum_b C_{ab}(f_b,f_a)$$

where f_a is the density in the $\boldsymbol{x}-\boldsymbol{v}$ phase space, where $\boldsymbol{x}$ is the position, $\boldsymbol{v}$ is the velocity, t is the time, and $\boldsymbol{A}=\boldsymbol{A}(\boldsymbol{x},\boldsymbol{v},t)$ is the acceleration due to electric and magnetic fields. Each C_{ab} is a Landau–Fokker–Planck collision operator, which describes Coulomb collisions with species b, and is given by

$$C_{ab}(f_b,f_a)=\gamma_{ab}m_b\frac{\partial}{\partial \boldsymbol{v}}\cdot\int d\boldsymbol{v}'\overleftrightarrow{U}(\boldsymbol{v}-\boldsymbol{v}')\cdot\left(\frac{1}{m_b}\frac{\partial}{\partial \boldsymbol{v}'}-\frac{1}{m_a}\frac{\partial}{\partial \boldsymbol{v}}\right)f_a(\boldsymbol{v})f_b(\boldsymbol{v}')$$

where

$$\begin{aligned}\overleftrightarrow{U}(\boldsymbol{v}-\boldsymbol{v}')&=\frac{1}{u}(\overleftrightarrow{I}-\hat{\boldsymbol{u}}\hat{\boldsymbol{u}})\\ \boldsymbol{u}&=\boldsymbol{v}-\boldsymbol{v}'\\ \gamma_{ab}&=2\pi\frac{\ln\Lambda_{ab}}{m_am_b}\frac{q_a^2}{4\pi\epsilon_0}\frac{q_b^2}{4\pi\epsilon_0}\\ \Lambda_{ab}&=\frac{r_{\max}}{r_{\min}};\quad r_{\max}=\lambda_{\mathrm{D}}\end{aligned}$$

where m_a and q_a denote the mass and charge of species a, $r_{\max}$ is an effective screening length typically taken to be the Debye length, and $r_{\min}$ is a "distance of closest approach." For classical Coulomb collisions, this is typically taken to be the distance at which the thermal and electrostatic energies of colliding particles balance. The notation can be made more compact by suppressing the velocity arguments where this is unambiguous, for example,

$$\begin{aligned}\psi&=\psi(\boldsymbol{v})\\ \psi'&=\psi(\boldsymbol{v}')\\ \overleftrightarrow{U}&=\overleftrightarrow{U}(\boldsymbol{v}-\boldsymbol{v}')\end{aligned}$$

to write

$$C_{ab}(f_b,f_a)=\gamma_{ab}m_b\frac{\partial}{\partial \boldsymbol{v}}\cdot\int d\boldsymbol{v}'\overleftrightarrow{U}\cdot\left(\frac{1}{m_b}\frac{\partial}{\partial \boldsymbol{v}'}-\frac{1}{m_a}\frac{\partial}{\partial \boldsymbol{v}}\right)f_af_b'.$$

The Landau–Fokker–Planck collision operator has key properties that are essential for connection to the fluid (collisional) limit(s). Like the Boltzmann operator, it describes elastic collisions and so conserves number density, momentum, and energy. It also satisfies an "H-theorem": $\dot{H}\geq 0$, where

$$H=S\equiv-\sum_a\int d\boldsymbol{v}\,f_a\ln f_a$$

is Boltzmann's kinetic entropy.

With the densities, total momentum, and energy constrained, H is maximum and $\dot{H}=0$ if and only if all species are Maxwellian and in equilibrium, i.e., they have the same temperatures and no relative flow.

A very useful form was provided by Rosenbluth *et al.* [20]

$$C_{ab}(f_b,f_a)=2\gamma_{ab}\frac{\partial}{\partial \boldsymbol{v}}\cdot\left[\left(\left[\frac{m_a+m_b}{m_b}\right]\frac{\partial H_{ab}}{\partial \boldsymbol{v}}f_a\right)-\frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\frac{\partial^2 G_{ab}}{\partial \boldsymbol{v}\partial \boldsymbol{v}}f_a\right)\right]$$

where the "Rosenbluth potentials" satisfy

$$\begin{aligned}\nabla_v^2H_{ab}&=-4\pi f_b\\ \nabla_v^2G_{ab}&=2H_{ab}\end{aligned}$$

This form is correct (in any coordinates) in 3-D velocity space and is the basis for direct implementations of the bilinear operator [1], [3], [4].

III. Linearized Operator

The preexisting implementation of Coulomb collisions in LOKI is a like-species operator based on linearization about a "background" Maxwellian. This is a major simplification because the Rosenbluth potentials and the drag and diffusion coefficients derived from them have easily calculated analytical expressions that depend only on the magnitude of a normalized kinetic velocity in the flow frame of the field-particle species. They can, therefore, be computed using a 1-D lookup table. For like species, the Maxwellian background is inherently in collisional equilibrium, i.e., it does not evolve due to collisions. The operator also uses a "reduced" backreaction term, which maintains conservation of momentum and energy and satisfies an H-theorem for the second-order entropy (expansion of H to second order in the departure from Maxwellian). For our application of interest, plasmas with multiple charged particle streams, we need to treat interstream collisions. In that case, the set of Maxwellians that approximate the distribution functions of the streams is no longer in collisional equilibrium, and this necessitates a significant generalization of the existing LOKI operator.

In general, the linearization can be done about any appropriate "background" distribution $F_{\mathrm{B}a}(\boldsymbol{v})$. Here, the subscript "B" stands for "Background," and "a" is the species subscript. We formally separate the distribution function as

$$f_a(\boldsymbol{v})=F_{\mathrm{B}a}(\boldsymbol{v})+\delta f_a(\boldsymbol{v}).$$

Inserting this into the collision operator (above) results in the separation of the collision operator into a "background" (0)

part and parts that are linear (1) and quadratic (2) in δf_a

$$\begin{aligned} C_{ab} &= C_{ab}^{(0)} + C_{ab}^{(1)} + C_{ab}^{(2)} \\ C_{ab}^{(0)} &\equiv C_{ab}(F_{\mathrm{B}b}, F_{\mathrm{B}a}) \\ C_{ab}^{(1)} &\equiv C_{ab}(F_{\mathrm{B}b}, \delta f_a) + C_{ab}(\delta f_b, F_{\mathrm{B}a}) \\ C_{ab}^{(2)} &\equiv C_{ab}(\delta f_b, \delta f_a). \end{aligned} \tag{II.1}$$

Linearization corresponds to neglecting $C_{ab}^{(2)}$, which is valid if $C_{ab}^{(2)} \ll C_{ab}^{(1)}$. Roughly speaking, this holds if $\delta f_a(\boldsymbol{v}) \ll F_{\mathrm{B}a}(\boldsymbol{v})$, but the latter may not always be a necessary condition, and in practice, we find that this is a good assumption even when the condition does not hold.

The linearized collision operator, i.e., the part that is linear in the perturbations $(\delta f_a, \delta f_b)$ is

$$C_{ab}^{(1)} = \gamma_{ab} m_b \frac{\partial}{\partial \boldsymbol{v}} \cdot \int d\boldsymbol{v}' \overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) \left(F'_{\mathrm{B}b} \delta f_a + \delta f'_b F_{\mathrm{B}a} \right).$$

A. Linearization About Isotropic Maxwellians, Possibly With Different Temperatures and Center Frames (Velocities)

The most useful approximate background state about which to linearize is a Maxwellian. The choice of number density, center (flow) velocity, temperature is not unique and needs to be specified. For intraspecies/self-collisions (e.g., of species a with a), this is

$$F_{\mathrm{M}a}(\boldsymbol{v}) \equiv n_a \left(\frac{m_a}{2\pi T_a} \right)^{d/2} \exp\left[-\frac{m_a (\boldsymbol{v} - \boldsymbol{V}_a)^2}{2T_a} \right] \tag{II.2}$$

where d is the velocity-space dimensionality and a reasonable (and usually optimal) choice of the moment quantities is $n_a = N[f_a]$, $\boldsymbol{V}_a = \boldsymbol{\Gamma}[f_a]/N[f_a]$, $T_a = m_a(K[f_a]/N[f_a] - (2/d)V_a^2)$, where

$$\begin{aligned} N[f] &\equiv \int d\boldsymbol{v}\, f \\ \boldsymbol{\Gamma}[f] &\equiv \int d\boldsymbol{v}\, \boldsymbol{v} f \\ K[f] &= \frac{2}{d} \int d\boldsymbol{v}\, v^2 f. \end{aligned} \tag{II.3}$$

The existing LOKI implementation of the collisions works in terms of the perturbation normalized to its Maxwellian (for that species or stream)

$$\delta \hat{f}_a \equiv \delta f_a / F_{\mathrm{M}a}(\boldsymbol{v})$$

in terms of which, the linearized collision operator can be written as

$$C_{ab}^{(1)} = \gamma_{ab} m_b \frac{\partial}{\partial \boldsymbol{v}} \cdot \int d\boldsymbol{v}' \overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) \left[F_{\mathrm{M}a} F'_{\mathrm{M}b} \left(\delta \hat{f}_a + \delta \hat{f}_b{}' \right) \right].$$

If $F_{\mathrm{M}a}$ and $F_{\mathrm{M}b}$ have the same temperature and center velocity, then

$$\overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) F_{\mathrm{M}a} F'_{\mathrm{M}b} = 0 \tag{II.4}$$

so that $C_{ab}^{(0)} = 0$ and

$$C_{ab}^{(1)} = \gamma_{ab} m_b \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[F_{\mathrm{M}a} \int d\boldsymbol{v}'\, F'_{\mathrm{M}b} \overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial \delta \hat{f}_b{}'}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial \delta \hat{f}_a}{\partial \boldsymbol{v}} \right) \right] \tag{II.5}$$

$C_{ab}^{(1)} = 0$ if

$$\frac{1}{m_b} \frac{\partial \delta \hat{f}_b}{\partial \boldsymbol{v}} = \frac{1}{m_a} \frac{\partial \delta \hat{f}_a}{\partial \boldsymbol{v}} = \boldsymbol{c}_1 + c_2 \boldsymbol{v}$$

That is, if

$$\delta f_s = \left(c_{0,s} + m_s \boldsymbol{c}_1 \cdot \boldsymbol{v} + \frac{1}{2} m_s c_2 v^2 \right) F_{\mathrm{M}s}$$

for species index $s = a, b$. Note that the constant $c_{0,s}$ can depend on the species, while $\boldsymbol{c}_1$ and c_2 must be the same for $s = a$ and b.

IV. Extension to Nonequilibrium Background State

When the background is not itself in collisional equilibrium, the above-mentioned approach would lead to additional terms. A simple (and important) example is when there are two streams with different center velocities and different temperatures. In this case

$$\left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) F_{\mathrm{M}a} F'_{Mb} = \left(\frac{\boldsymbol{v} - \boldsymbol{V}_a}{T_a} - \frac{\boldsymbol{v}' - \boldsymbol{V}_b}{T_b} \right) F_{\mathrm{M}a} F'_{\mathrm{M}b}.$$

This shows that some generalization of the LOKI collision implementation is needed for interstream collisions.

We can adapt the approach used for the collision implementation in LOKI to the case of multiple streams by noting that the effect of collisions of species a with a Maxwellian species b is to drive the distribution function of a toward a Maxwellian (for species of mass m_a) with the temperature and center velocity of the species-b Maxwellian

$$F_{\mathrm{M}ab}(\boldsymbol{v}) \equiv n_a \left(\frac{m_a}{2\pi T_b} \right)^{3/2} \exp\left[-\frac{m_a (\boldsymbol{v} - \boldsymbol{V}_b)^2}{2T_b} \right].$$

To isolate this effect in the forward scattering (drag-diffusion) term, note that if $\delta f_a = \alpha F_{\mathrm{M}ab}$, where α is the independent of $\boldsymbol{v}$, then

$$\begin{aligned} C_{ab}(F_{\mathrm{M}b}, \delta f_a) &= \alpha C_{ab}(F_{\mathrm{M}b}, F_{\mathrm{M}ab}) \\ &= 0. \end{aligned}$$

Similarly for the backreaction term, we note that if $\delta f_b = \beta F_{\mathrm{M}ba}$, where β is the independent of $\boldsymbol{v}$, then

$$C_{ab}(\delta f_b, F_{\mathrm{M}a}) \equiv \beta C_{ab}(F_{\mathrm{M}ba}, F_{\mathrm{M}a}) = 0.$$

These results follow from

$$\begin{aligned} &\overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) F_{\mathrm{M}ab} F'_{\mathrm{M}b} \\ &\qquad = \overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) F_{\mathrm{M}a} F'_{\mathrm{M}ba} = 0. \end{aligned}$$

Using the notation

$$\delta \hat{f}_{ab} \equiv \delta f_a / F_{\mathrm{M}ab}(\boldsymbol{v})$$

the linear part of the collision operator can be written as

$$\begin{aligned} C_{ab}^{(1)} &= \gamma_{ab} m_b \frac{\partial}{\partial \boldsymbol{v}} \cdot \int d\boldsymbol{v}' \overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) \\ &\quad \times \left[F_{\mathrm{M}ab} F'_{\mathrm{M}b} \delta \hat{f}_{ab} + F_{\mathrm{M}a} F'_{\mathrm{M}ba} \delta \hat{f}'_{ba} \right] \\ &= \gamma_{ab} \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[F_{\mathrm{M}a} \left(\int d\boldsymbol{v}' \, F'_{Mba} \overleftrightarrow{U} \cdot \frac{\partial \delta \hat{f}'_{ba}}{\partial \boldsymbol{v}'} \right) \right] \\ &\quad - \gamma_{ab} \frac{m_b}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[F_{\mathrm{M}ab} \left(\int d\boldsymbol{v}' \, F'_{\mathrm{M}b} \overleftrightarrow{U} \right) \cdot \frac{\partial \delta \hat{f}_{ab}}{\partial \boldsymbol{v}} \right]. \end{aligned}$$

This generalizes (II.5). Defining the collisional diffusion tensor

$$\overleftrightarrow{D}_{ab} \equiv \overleftrightarrow{D}_{ab}(\boldsymbol{v}) \equiv \gamma_{ab} \frac{m_b}{m_a} \int d\boldsymbol{v}' F'_{\mathrm{M}b} \overleftrightarrow{U}$$

and adding in the background term, we can write

$$\begin{aligned} &C_{ab}^{(0)} + C_{ab}^{(1)} \\ &\quad = -\frac{\partial}{\partial \boldsymbol{v}} \cdot \left[F_{\mathrm{M}ab} \overleftrightarrow{D}_{ab}(\boldsymbol{v}) \cdot \frac{\partial \hat{f}_{ab}}{\partial \boldsymbol{v}} \right] \\ &\qquad + \gamma_{ab} \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[F_{\mathrm{M}a} \left(\int d\boldsymbol{v}' \, F'_{\mathrm{M}ba} \overleftrightarrow{U} \cdot \frac{\partial \delta \hat{f}'_{ba}}{\partial \boldsymbol{v}'} \right) \right]. \end{aligned} \quad \text{(III.1)}$$

Note that this is an exact expression for $C_{ab}^{(0)} + C_{ab}^{(1)}$. No further approximations have been made beyond the linearization. We will refer to the first term, which is entirely a diffusion operator acting on $\hat{f}_{ab}$ as the "forward" diffusion term and the second integrodifferential term as the "backreaction" term. The backreaction term enforces momentum and energy conservation. This form is useful both in that it tells how to implement the forward term using preexisting code machinery, such as that in LOKI code, and because of its clear intuitive interpretation. The forward term becomes zero if $(\partial \hat{f}_{ab}/\partial \boldsymbol{v}) = 0$, i.e., if $f_a = \alpha F_{\mathrm{M}ab}$ for any α independent of $\boldsymbol{v}$. The backreaction term becomes zero if $(\partial \delta \hat{f}_{ba}/\partial \boldsymbol{v}) = 0$, i.e., if $\delta f_b = \beta F_{\mathrm{M}ba}$ for any β independent of $\boldsymbol{v}$.

V. Implementation Considerations

A. Issue Due to LOKI 2-D Cartesian Velocity Space

There is a significant issue in the LOKI implementation concerning the mixing of 2-D and 3-D velocity spaces. In order to have the correct asymptotic dependences of the Rosenbluth potentials on the velocity arguments and the thermal velocities, these must be calculated for a 3-D velocity-space distribution function (Maxwellian in our case). However, LOKI uses a 2-D Cartesian velocity space. The derivation of the momentum- and energy-conservation relations for the linearized operator requires an interchange of the orders of two velocity-space integrations. This interchange can be made if the velocity spaces over which these integrations are done are the same. However, the mixture of velocity spaces necessary in the LOKI implementation has the consequence that these changes in the order of integration are no longer available, that is,

$$\begin{aligned} &\int d\boldsymbol{v} \begin{pmatrix} m_a \boldsymbol{v} \\ \frac{1}{2} m_a v^2 \end{pmatrix} C_{ab}(f_b, f_a) \\ &\quad = \gamma_{ab} m_b \int d^2\boldsymbol{v} \begin{pmatrix} m_a \boldsymbol{v} \\ \frac{1}{2} m_a v^2 \end{pmatrix} \frac{\partial}{\partial \boldsymbol{v}} \\ &\qquad \cdot \int d^3\boldsymbol{v}' \overleftrightarrow{U} \cdot \left(\frac{1}{m_b} \frac{\partial}{\partial \boldsymbol{v}'} - \frac{1}{m_a} \frac{\partial}{\partial \boldsymbol{v}} \right) f_a f'_b \\ &\quad \neq -\int d\boldsymbol{v} \begin{pmatrix} m_b \boldsymbol{v} \\ \frac{1}{2} m_b v^2 \end{pmatrix} C_{ba}(f_a, f_b). \end{aligned}$$

Thus, the raw full (and linearized) Coulomb collision operators no longer satisfy momentum and energy conservation with this mixture of phase spaces. Fortunately, an appropriate modified backreaction term can restore conservation of energy and momentum, and there exists reduced-rank version of such an operator.

B. Approximate Backreaction Operator

The "raw" back reaction operator of (III.1)

$$C_{ab}(f_b, F_{\mathrm{B}a}) = \gamma_{ab} \frac{\partial}{\partial \boldsymbol{v}} \cdot \left[F_{\mathrm{M}a} \left(\int d\boldsymbol{v}' \, F'_{\mathrm{M}ba} \overleftrightarrow{U} \cdot \frac{\partial \hat{f}'_{ba}}{\partial \boldsymbol{v}'} \right) \right]$$

is an integral operator on f_b of high rank. Reduced-rank "model" backreaction operators have been developed for the case of equilibrium backgrounds [17]–[19]. These preserve the conservation of momentum and energy, Galilean invariance and the H-theorem in the case of equilibrium backgrounds. While they have so far been used mostly in explicit settings to make the calculation of the operator computationally cheaper, the reduction in rank may also result in a significant speedup for implicit calculations.

We have generalized this reduced backreaction operator to allow for nonequilibrium backgrounds. The formal expression for this operator is

$$\begin{aligned} &C_{ab\mathrm{BR}}(f_b, F_{Ma}) \\ &\quad = -\frac{m_b}{m_a} \\ &\qquad \times \left[\frac{\langle \boldsymbol{v}_a | C_{ba}(F_{\mathrm{M}a}, f_b) \rangle}{\frac{1}{d} \langle \boldsymbol{v}_a | \cdot C_{ab}(F_{\mathrm{M}ba}, F_{\mathrm{M}a} \boldsymbol{v}_a) \rangle} \cdot C_{ab}(F_{\mathrm{M}ba}, F_{\mathrm{M}a} \boldsymbol{v}_a) \right. \\ &\qquad\quad \left. + \frac{\langle \boldsymbol{v}_a^2 | C_{ba}(F_{\mathrm{M}a}, f_b) \rangle}{\langle \boldsymbol{v}_a^2 | C_{ab}\left(F_{\mathrm{M}ba}, F_{\mathrm{M}a} \boldsymbol{v}_a^2\right) \rangle} C_{ab}\left(F_{\mathrm{M}ba}, F_{\mathrm{M}a} \boldsymbol{v}_a^2\right) \right] \end{aligned}$$

where

$$\langle g | h \rangle \equiv \int d\boldsymbol{v} \, g(\boldsymbol{v}) h(\boldsymbol{v}).$$

One can think of this as a reduction of rank via representation of $C_{ab}(f_b, F_{Ma})$ as sum of terms $\propto C_{ab}(F_{\mathrm{M}ba} \boldsymbol{v}_a^n, F_{\mathrm{M}a})$, where $\boldsymbol{v}_a \equiv \boldsymbol{v} - \boldsymbol{V}_a$, and with coefficients such that momentum and energy are conserved. These terms can then be expressed as forward operators via use of identities such as

$$C_{ab}\left(\begin{bmatrix} \boldsymbol{v} \\ v^2 \end{bmatrix} F_{\mathrm{M}ba}, F_{\mathrm{M}a} \right) = -\frac{m_a}{m_b} C_{ab}\left(F_{\mathrm{M}ba}, F_{\mathrm{M}a} \begin{bmatrix} \boldsymbol{v} \\ v^2 \end{bmatrix} \right).$$

It can be shown directly that this form of the backreaction operator enforces momentum and energy conservation even in a case, such as ours, where the underlying operator $C_{ab}(f_b, f_a)$ would not.

C. Choices of Moments to Use for Maxwellians

As stated earlier, we will choose each of the background distributions, used in the calculation of the Rosenbluth potentials, to be a Maxwellian as given by (II.2). The primary choice expected to be used for the moments is as given in (II.3). That is, given f (or f_B and δf, so that $f = f_\mathrm{B} + \delta f$) before collisions, take moments as specified by (II.3) of $f \to N[f]$, $\mathbf{\Gamma}[f]$, $K[f]$, base the Maxwellian on these, and choose $\delta f = f - F_\mathrm{M}$. Then, δf has zero density, momentum, and energy before collision. Collisions may impart flow or energy to δf. The backreaction terms then enforce conservation of momentum and energy between species within perturbed parts separately. On next step (or RK stage), this projection removes all momentum and energy from δf and puts it into F_M.

A second option, which is useful for verification of the backreaction terms with nonequilibrium background, is to use the moments of f_B. That is, use $N[f_\mathrm{B}]$, $\mathbf{\Gamma}[f_\mathrm{B}]$, $K[f_\mathrm{B}]$, and base the Maxwellian on these. The new precollision perturbed distribution function is

$$\delta\hat{f} = \delta f + f_\mathrm{B} - F_\mathrm{M}.$$

$\delta\hat{f}$ can have nonzero density, momentum, and energy before collision. This projection puts the nonmaxwellian part of f_B into $\delta\hat{f}$. Then, apply collisions separately to F_M resulting in the new f_B, and to $\delta\hat{f}$ (including backreaction term). The backreaction terms enforce conservation of momentum and energy between species within background and perturbed parts separately. An additional suboption, again useful for verification, is $\delta\hat{f} = \delta f$, i.e., neglect the kinetic piece $f_\mathrm{B} - F_\mathrm{M}$ caused by background collisions in $\delta\hat{f}$.

VI. Tests of the Collision Operator

Demonstrations of momentum and energy conservation test the backreaction terms, including density and mass factors. The absence of evolution of two shifted-Maxwellian streams in any frame is a key demonstration of Galilean invariance. The entropy evolution can be tracked to show that it increases, and so does not violate the H-theorems satisfied by the full nonlinear operator and by the linearized operator in the case where the backgrounds are in equilibrium. In the general case, we have not obtained an H-theorem, so there is a possibility that the entropy (either the "full" entropy or with terms cubic and higher in δf neglected) might decrease. However, any decrease should not persist in undriven situations where collisional equilibration is expected.

We would like to verify that the collision operator gives correct results for relaxation rates, for example, those for relaxation of modest flow differences and modest temperature differences. These rates are valuable tests of the forward terms. For the "physical" case of 3-D velocity spaces, results for these rates are given in [22] and [23]. For test particle species a colliding with Maxwellian b, the 3-D relaxation rate for both temperature and momentum is

$$\nu_{ab} = \frac{4\sqrt{2\pi}}{3} n_b q_a^2 q_b^2 \left(\frac{\sqrt{\mu_{ab}}}{m_a}\right)\left(\frac{T}{m_b}\right)^{-3/2}. \qquad \text{(V.1)}$$

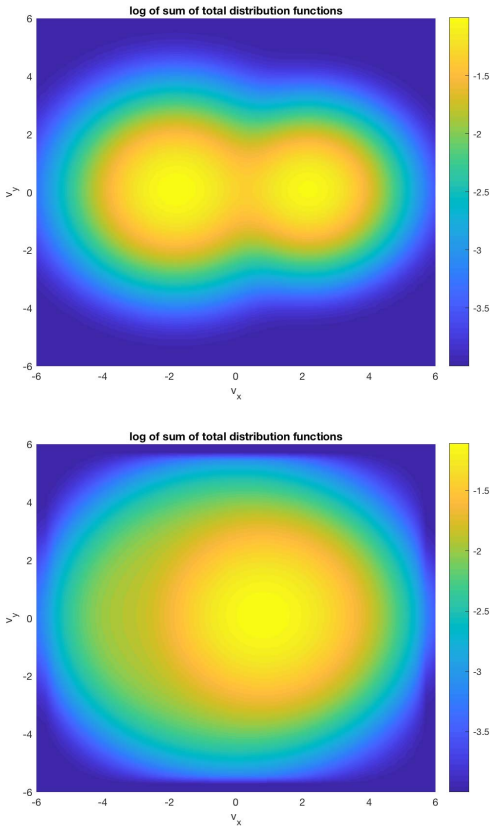

Fig. 1. Evolution of (the logarithm of) the total velocity-space density for a two-stream collision test case. The first frame shows the initial condition and the second frame shows a late time state.

A. Verification Issue Due to Mixed 2-D and 3-D Velocity Spaces

The present operator is targeted to LOKI which uses a 2-D Cartesian velocity space. This can be thought of as the $v_z = 0$ subspace of the "physical" 3-D (v_x, v_y, v_z) space. The resulting 2-D moment integrals taken to get the rates are fundamentally different from those in 3-D. Thus, one cannot verify relaxation rates by comparison with published 3-D results such as those that in [22] and [23]. One would need to work out analogous 2-D rates (and with the 3-D velocity space-based Rosenbluth potentials used in the implementation). An independent calculation of such rates is presently unavailable. Thus, the main available test is that such rates are "reasonable," that is have values similar to those for the 3-D–3-D case. Differences in physics for this 2-D operator and the "full" 3-D operator are also expected to be present for kinetic effects.

B. Multiple Feature Test Case

We have implemented the above operator for the 2-D Cartesian V space in a testbed (MATLAB) code, extending a

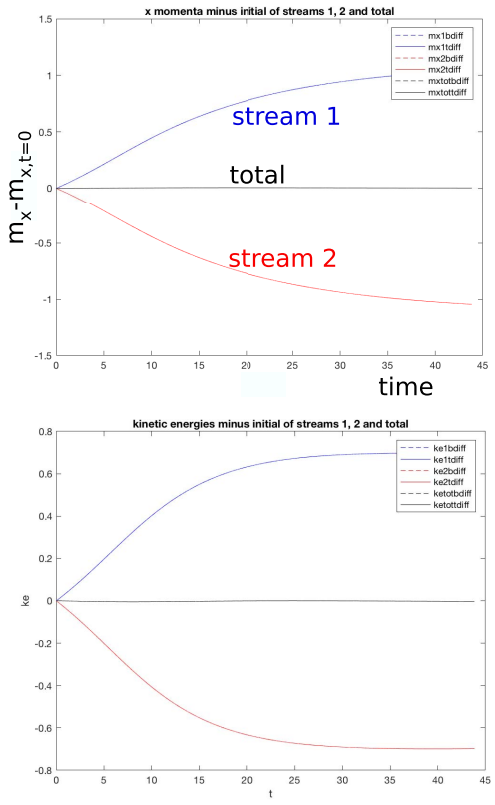

Fig. 2. Evolution of stream (blue and red) and total (black) momenta and kinetic (thermal + flow) energies for the two-stream collision test case. There are dashed curves, which are those calculated for the background Maxwellians only, and solid curves for the total distribution functions.

previous implementation for single like-species case and have carried out verification cases run for various setups. A general setup that simultaneously tests all of the aspects of the operator other than the quantitative relaxation rates consists of two ion streams with different masses, densities, flow velocities (with no particular centering about the velocity-space origin), and temperatures. Demonstration of conservation of momentum and energy for such a case simultaneously checks mass, and density factors, and Galilean invariance.

Fig. 1 shows the evolution of the total velocity-space density for such a two-stream collision test case. Only interstream collisions are included. The distribution function is seen to evolve toward a Maxwellian with no relative flow between the streams, and with a center not at the origin, consistent with momentum conservation.

Fig. 2 shows the evolution of individual stream (blue and red) and total (black) momenta and kinetic (thermal + flow) energies for the two-stream collision test case. There are dashed curves, which are those calculated for the background Maxwellians only, and solid curves for the total distribution functions. Because of the particular choice (first option) of the moments used for the Maxwellian (i.e., those of the total distribution function, the dashed curves are indistinguishable from the solid ones and are, therefore, "invisible" in these plots. These plots show damping rates qualitatively consistent with (V.1) and excellent conservation of the total momentum and energy.

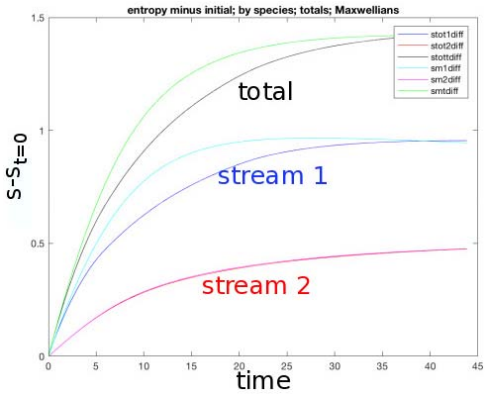

Fig. 3. Evolution of entropy components for the two-stream collision test case. Shown are stream-component (blue and red) and total (black) entropies based on the full (f) distribution functions, as well as the stream-component (cyan and magenta) and total (green) entropies based on the Maxwellian background (F_M) distribution functions.

Fig. 3 shows the evolution of stream (blue and red) and total (black) entropy components based on the full (f) distribution functions, for the two-stream collision test case, as well as the stream (cyan and magenta) and total (green) entropy components, based on the Maxwellian background (F_M) distribution functions. The key result from this figure is that the total entropy (black curve) is monotonically increasing, consistent with the H-theorem, even if strictly speaking these do not apply to the present operator.

VII. Conclusion

We have developed a Coulomb collision operator suitable for the simulation of multiple interpenetrating plasma streams in an Eulerian kinetic code framework. This operator is based on previously developed linearized operators but represents a significant extension due to the nonequilibrium background state about the linearization is done and due to the necessarily disparate (2-D and 3-D) velocity spaces used in the particular (LOKI) code representation and in underlying calculation of the drag and diffusion coefficients (Rosenbluth potentials).

This operator has a simplifying aspect in that it is based on the calculation of the drag and diffusion coefficients using a Maxwellian approximation to the distribution functions. The reduced backreaction operator developed here has the fortunate property that it fixes nonconservation issues that result from the disparate 2-D–3-D velocity spaces used.

This operator has been implemented in a testbed (MATLAB) code, and key verification tests have been carried out and presented. The operator, as implemented shows excellent (essentially perfect) momentum and energy conservation and monotonic entropy increase for the test problem presented, which is a quite general test that combines several differences between the interacting plasma streams.

It would be desirable to establish a clean H-theorem for this operator. While we have been able to obtain various illustrative forms of the entropy evolution equation, such a clean H-theorem has not been derived. However, the monitoring of the entropy evolution in the test cases that we have carried out does not show any entropy decrease, and such entropy

monitoring can be carried out in actual applications to gain further understanding and to watch for possible problems in this regard.

We believe that his approach will be useful in other settings, e.g., in 3-D velocity space with axisymmetry (using cylindrical coordinates in velocity space) or, when the computational capability eventually becomes available to make it useful, in full 3-D velocity-space simulations.

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